Since the "quarter-wavelength transformer" principle operates for low-loss line sections that are *any odd number* of quarter-wavelengths long, another possible solution of Example 7.7 would be to use a $4\frac{1}{4}$ wavelength section of the line whose characteristic impedance is 134 ohms. The length of this section would be about 101.5 ft. In practice, however, this solution would be a poor one, since actual signals always occupy a finite range of frequencies. Over any small frequency interval the range of variation of the term βl in (*7.20*) for a $4\frac{1}{4}$ wavelength line section would be 17 times as great as for a $\frac{1}{4}$ wavelength line section. In the solution illustrated by Fig. 7-2, the 95 ft of line with characteristic impedance 500 ohms would have little effect on the frequency sensitivity of the system, because this characteristic impedance is equal to the impedance of the source to which the line is connected. These statements can easily be checked quantitatively by using the transmission-line circle diagram discussed in Chapter 9.

It has been pointed out in Section 2.1 that the uniformity postulate underlying the transmission line analysis of this book is violated in the vicinity of terminations and other discontinuities on transmission lines. In the use of quarter-wavelength or half-wavelength transformers, such discontinuities occur at each end, either where the line section is connected to a source or load, or where it is connected to a transmission line section of different characteristic impedance. The main practical consequence of this departure from idealized conditions is that the optimum length of these transformers in specific applications is likely to differ slightly from the value calculated using the equations of uniform lines. The difference is generally much less than one transverse dimension of the line, and the experimentally optimum length is best found by starting from the calculated value and making small adjustments.

7.5. Determination of transmission line characteristics from impedance measurements.

When an arbitrary length of any general transmission line is terminated in an open circuit or a short circuit, its input impedance is determined completely by the propagation factors α and β, the characteristic impedance Z_0 and the line length l. From (*7.20*), if $Z_T = 0$ (but $\alpha \neq 0$) the input impedance Z_{sc} of a line of length l with short circuit termination is

$$Z_{sc} = Z_0 \tanh(\alpha + j\beta)l \qquad (7.25)$$

The input impedance Z_{oc} of the same line with open circuit termination is

$$Z_{oc} = Z_0 \coth(\alpha + j\beta)l \qquad (7.26)$$

If Z_{sc} and Z_{oc} are measured at the same frequency, for a line section of length l, then Z_0, α, β, and l will have the same values in both of the equations (*7.25*) and (*7.26*). Multiplying together the corresponding sides of these equations gives

$$Z_0 = \sqrt{Z_{sc} Z_{oc}} \qquad (7.27)$$

This is a valuable and universally valid equation by which the characteristic impedance of any type of uniform transmission line can be obtained from two impedance measurements made on a sample length of the line, using two readily available terminal load impedances.

Two precautions must be observed in making the measurements needed for this calculation. First, the impedance-measuring device must be capable of measuring "balanced" impedances if the line conductors are symmetrical (e.g. a parallel-wire line or a shielded pair), or of measuring "unbalanced" impedances if the line has one of its two conductors acting as a shield or "ground" (e.g., a coaxial line or a stripline). Second, the length of line l cannot be completely arbitrary but must be chosen so that both Z_{sc} and Z_{oc} have values appropriate to the impedance-measuring device. It is obvious, for example, that for an extremely short section of any line Z_{sc} might be too small and Z_{oc} too large to be accurately measurable by any available bridges. Use of the transmission line charts discussed in

Chapter 9 will show that line-section lengths close to any odd number of eighths of a wavelength are particularly appropriate. For such line lengths Z_{sc}, Z_{oc} and Z_0 will all have similar magnitudes. If the wavelength is only approximately known, measurements can be made at several values of l until this condition is found.

The attenuation factor α and the phase propagation factor β can also be calculated from the measured impedances Z_{sc} and Z_{oc}. Dividing corresponding sides of *(7.25)* by those of *(7.26)*,

$$\sqrt{Z_{sc}/Z_{oc}} = \tanh(\alpha + j\beta)l$$

Expanding this hyperbolic tangent in exponential form, using $\gamma = \alpha + j\beta$,

$$\sqrt{Z_{sc}/Z_{oc}} = (1 - e^{-2\gamma l})/(1 + e^{-2\gamma l})$$

which gives

$$e^{2\gamma l} = \frac{1 + \sqrt{Z_{sc}/Z_{oc}}}{1 - \sqrt{Z_{sc}/Z_{oc}}}$$

Taking logarithms of both sides,

$$(\alpha + j\beta)l = \tfrac{1}{2}\log_e \frac{1 + \sqrt{Z_{sc}/Z_{oc}}}{1 - \sqrt{Z_{sc}/Z_{oc}}}$$

The logarithm of a complex number expressed in polar form $Ae^{j\phi}$ is defined by

$$\log_e Ae^{j\phi} = \log_e A + j(\phi + 2n\pi)$$

The attenuation factor α is therefore given by

$$\alpha = \frac{1}{2l}\log_e \left|\frac{1 + \sqrt{Z_{sc}/Z_{oc}}}{1 - \sqrt{Z_{sc}/Z_{oc}}}\right| \quad \text{nepers/m} \tag{7.28}$$

when l is in meters. The phase propagation factor β is given by

$$\beta = \frac{1}{2l}\left\{\left(\text{phase angle of } \frac{1 + \sqrt{Z_{sc}/Z_{oc}}}{1 - \sqrt{Z_{sc}/Z_{oc}}}\right) + 2n\pi\right\} \quad \text{radians/m} \tag{7.29}$$

This method does not determine a unique value for β, but a series of values differing consecutively by π/l rad/m. In a practical case it may sometimes be difficult to decide which value in the series is the correct answer.

Example 7.8.

At a frequency of 20.0 megahertz the input impedance of a section of flexible coaxial transmission line 32.0 m long is measured, first with the line terminated in a short circuit and then with the line terminated in an open circuit. The respective values obtained are $Z_{sc} = 17.0 + j19.4$ ohms and $Z_{oc} = 115 - j138$ ohms. Find the attenuation factor, the phase propagation factor, and the characteristic impedance of the line.

The impedances are needed in polar form for all of the calculations. $Z_{sc} = 25.7\underline{/48.8^\circ}$ and $Z_{oc} = 179\underline{/-50.2^\circ}$ ohms. The characteristic impedance from *(7.27)* is then $Z_0 = \sqrt{Z_{sc}Z_{oc}} = 68\underline{/-0.7^\circ}$ ohms.

For determining α and β the quantity $\sqrt{Z_{sc}/Z_{oc}} = 0.378\underline{/49.5^\circ} = 0.245 + j0.288$ is required. Using equation *(7.28)*,

$$\alpha = \frac{1}{2(32.0)}\log_e \left|\frac{1.245 + j0.288}{0.755 - j0.288}\right| = 0.0072 \quad \text{nepers/m}$$

The phase angle of the term $(1.245 + j0.288)/(0.755 - j0.288)$ is found to be 33.9°. From equation *(7.29)*, $\beta = (0.59 + 2n\pi)/(2 \times 32.0)$ rad/m, but there is no basis for choosing the value of n.

At the frequency of the measurements, the free space wavelength is 15.0 m and the corresponding value of β would be found from $\beta = 2\pi/\lambda = 0.419$. Since the line contains plastic dielectric it is expected that the wavelength on the line may be as much as 30% shorter than the free space value, but the figure is not known accurately. Hence β might conceivably lie between about 0.50 and 0.65. The above equation gives $\beta = 0.40$ for $n = 4$, $\beta = 0.50$ for $n = 5$, $\beta = 0.60$ for $n = 6$, and $\beta = 0.70$ for $n = 7$.

On the evidence available, there is no conclusive basis for choosing between the two intermediate values. The data indicates that the line section is between two and three wavelengths long. By making the same impedance measurements on a shorter length of line, lower values of n will occur in the equation for β and there will be less doubt about which value should be chosen.

For a section of the same line 1.50 m long, the impedance values measured were $Z_{sc} = 0 + j88$ ohms and $Z_{oc} = 0 - j52$ ohms, the resistive component in each case being less than 1 ohm. From these values the characteristic impedance is calculated as $68\underline{/0^\circ}$ ohms.

Because the quantity $\sqrt{Z_{sc} Z_{oc}} = 0 + j1.30$ is purely imaginary, the attenuation factor α is indicated as having value zero. The phase angle of the term $(1 + j1.30)/(1 - j1.30)$ is 105° or 1.83 rad. From equation (*7.29*), $\beta = (1.83 + 2n\pi)/(2 \times 1.50) = 0.61$ rad/m for $n = 0$, or 2.70 rad/m for $n = 1$. It is clear now that $n = 6$ gave the correct value in the previous data and that the measured value of β is 0.60 (or 0.61) rad/m. It is also evident that measurements on this short length of line cannot be used to obtain a value for the attenuation factor α.

Consideration of the equations shows that this method of determining α and β from measurements of Z_{sc} and Z_{oc} will give the best results for α when the line section has a total attenuation of about 3 db, and will give the least ambiguous results for β when the line is about one-eighth wavelength long. Except at frequencies in the kilohertz range, a single piece of any practical transmission line will not satisfy both of these conditions, so that measurements of α and β should usually be made on two different line sections, one much longer than the other. Measurements on either section will give satisfactory data for the determination of Z_0.

7.6. Complex characteristic impedance.

The characteristic impedance of a transmission line was originally defined in terms of the line's distributed circuit constants, and given by equation (*4.12*) as

$$Z_0 = \sqrt{(R + j\omega L)/(G + j\omega C)}$$

Since R, L, G, C and ω are all positive real numbers for a passive transmission line, it follows from this expression that the phase angle of Z_0 must lie between $-45°$ and $+45°$ or, if $Z_0 = R_0 + jX_0$, the ratio X_0/R_0 must lie between -1 and $+1$. The extreme values occur when *either* $R \gg \omega L$ and $G \ll \omega C$, *or* $R \ll \omega L$ and $G \gg \omega C$. For either of these sets of conditions the defining equation (*4.10*), $\alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$, shows that $\alpha = \beta$.

Noting that $\beta = 2\pi/\lambda$, the relation $\alpha = \beta$ has the physical meaning that the attenuation of the line is 2π *nepers per wavelength* or 54.6 decibels per wavelength. Transmission lines useful at high frequencies have attenuations per wavelength smaller than this value by several orders of magnitude, but Table 5.1, page 55, shows that a standard type of telephone cable-pair can have α very nearly equal to β (and $|X_0|$ nearly equal to R_0) at frequencies below about 1 kilohertz.

Neither a large value of attenuation factor α in nepers per meter, nor a large total attenuation αl in nepers, is a sufficient condition to ensure that the characteristic impedance of a transmission line will have a substantial phase angle. The attenuation over *one wavelength* of line, $\alpha\lambda$ nepers, must be large, and it must be caused predominantly by *one* of *R or G* and not by a combination of the two. It has already been noted that if the losses are due equally to R and G, Z_0 is real, no matter how high the losses are.

When the characteristic impedance of a transmission line has an appreciable phase angle, some peculiar results arise, which need further discussion. If, for example, $Z_0 = R_0 + jX_0$ and the line has a terminal load impedance $Z_T = 0 - jX_0$, the reflection coefficient determined from equation (*7.10*) is

$$\rho_T = (-jX_0 - R_0 - jX_0)/(-jX_0 + R_0 + jX_0) = -1 - j2X_0/R_0$$